Systems & Control Letters 123 (2019) 62–68

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Stabilization of discrete time stochastic system with input delay and control dependent noise ☆

Cheng Tan [a,b], Lin Yang [b], Fangfang Zhang [c], Zhengqiang Zhang [a,*], Wing Shing Wong [b]

[a] College of Engineering, QuFu Normal University, Rizhao, Shandong 276800, China
[b] Department of Information Engineering, The Chinese University of Hong Kong, Shatin, N. T., Hong Kong
[c] School of Electrical Engineering, Zhengzhou University, Zhengzhou 450001, China

ARTICLE INFO

Article history:
Received 1 September 2017
Received in revised form 1 November 2018
Accepted 8 November 2018
Available online 29 November 2018

Keywords:
Delay dependent Lyapunov equation (DDLE)
Input delay
Stabilization
Stochastic system

ABSTRACT

In this paper, we introduce a delay dependent Lyapunov equation (DDLE) approach to study the mean square stabilization for discrete time stochastic system with both input delay and control dependent noise. The innovative contributions of this paper are twofold. First, for a general stochastic system with input delay and multiplicative noises, we derive a necessary stabilizing condition based on a coupled Lyapunov equation (CLE). Second, we present a set of necessary and sufficient stabilizing conditions for the considered stochastic system. We show that the stochastic system is stabilizable is equivalent to that the DDLE has a positive definite solution. In this case, the constructed CLE is equivalent to the DDLE. Moreover, based on the Lyapunov stabilizing result, we further derive a spectrum stabilizing criterion. To confirm the effectiveness of our theoretic results, two illustrative examples are included.

1. Introduction

Since a class of practical models can be described as stochastic systems, stochastic control theory has attracted considerable interest in a variety of the engineering fields over the past several decades; See [1–10] for a partial list of references. Parallel to the control theory for deterministic systems, a great deal of control problems have been extended to stochastic models, such as stabilization, optimal filter, linear quadratic (LQ) optimization. In [5], the indefinite stochastic LQ optimization problem was investigated and the optimal control policy was developed via a unique stabilizing solution to a generalized algebraic Riccati equation (GARE). In [6], for stochastic system with multiplicative noise, a necessary and sufficient stabilizing condition was derived with operator spectrum theory, which is said to be the spectrum stabilizing criterion. Moreover, in [10], the Lyapunov stabilizing criterion was proposed in terms of the feasibility of a certain LMI, which can be verified by LMI solvers in MATLAB. Compared with Riccati type and spectrum type criteria, the Lyapunov stabilizing criterion is much convenient to verify.

Recently, networked control systems (NCSs) that exchange information between a plant and a remote controller through a shared communication network have been a topic of active research in both the academia and the industry; See [11–17] and the references therein. In an open communication network, signal transmission over a routing path invariably experiences transmission delay and packet dropout. These uncertainties may degrade the system performance and even destabilize the whole system. Generally speaking, it is common to model an integrated NCS with both transmission delay and packet dropout as a stochastic system with input delay and control dependent noise. Therefore, it is of great significance to study the stabilization problem of such stochastic systems. In [18], the stochastic LQ optimization and stabilization was considered for a class of discrete time stochastic systems involving input delay and multiplicative noises. The necessary and sufficient stabilizing condition was derived in terms of a unique positive definite solution to a coupled Riccati equation (CRE) with two variables, where the optimal and stabilizing control policy was shown to be the feedback of the conditional expectation of the state. However, it is quite difficult to compute the value of the positive definite solution satisfying the CRE. To the best of our knowledge, how to best utilize available information to design a stabilizing control policy as well as the search for the less conservative Lyapunov type stabilizing condition remain open questions.

In this paper, we focus on the mean square stabilization problem for a discrete time stochastic system with both input delay and control dependent noise. Our research methodology is described

☆ This work was supported in part by a grant from the Research Grants Council of the Hong Kong Special Administrative Region under Project GRF. 14200217, the National Natural Science Foundation of China under Grants 61803224, 61873330, the Taishan Scholarship Project of Shandong Province, China under Grant tsqn20161032, and Shandong Provincial Natural Science Foundation, China for Distinguished Young Scholars under Grant JQ201515.

* Corresponding author.
E-mail addresses: tancheng1987love@163.com (C. Tan), yl015@ie.cuhk.edu.hk (L. Yang), zhangff1986@163.com (F. Zhang), qufuzzq@126.com (Z. Zhang), wswong@ie.cuhk.edu.hk (W.S. Wong).

https://doi.org/10.1016/j.sysconle.2018.11.003
0167-6911/© 2018 Elsevier B.V. All rights reserved.

as follows. First, we consider a general stochastic system with input delay and multiplicative noises. Under the assumption of the mean square stabilization, we construct a CLE, which is the basis to derive the Lyapunov stabilizing criterion. Second, we derive a set of DDLE based conditions for stabilization. It is shown that the considered stochastic system is stabilizable if and only if the developed DDLE has a positive definite solution. Note that in this case the CLE is equivalent to the constructed DDLE. Moreover, on the basis of Lyapunov stabilizing criterion, we derive the spectrum stabilizing criterion, in which we demonstrate that the stochastic system is stabilizable if and only if the spectral radius is less than one. These stabilizing criteria are first obtained in the framework of stochastic system with both input delay and control dependent noise, which run in parallel to the classical stochastic stabilization results in [6,10].

Notations: Let A' and $\mathbf{Tr}(A)$ denote the transpose and the trace of matrix A. $A \geq 0 (> 0)$ means that A is a positive semidefinite (positive definite) matrix and $A \geq B (> B)$ means that $A - B \geq 0 (> 0)$. Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space and $\mathbb{S}^n$ be the space of all $n \times n$ positive semidefinite matrices. δ_{ts} is a Kronecker function. $\{w_t, t \in \mathbb{N}\}$ means a sequence of real random variables defined on the complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_t = \sigma\{w_s, s = 0, \ldots, t\}$. Define $\hat{x}_{t|s} = \mathbf{E}[x_t|\mathcal{F}_s]$ which signifies the conditional expectation of x_t w.r.t. $\mathcal{F}_s$.

2. Problem formulation

Without loss of generality, we consider the following discrete time stochastic system with input delay and control dependent noise:

$$x_{t+1} = Ax_t + Bu_{t-d} + \omega(t)Cu_{t-d}, \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state and $u_t \in \mathbb{R}^m$ is the input control with a constant input delay $d > 0$. We assume that $x_0, u_i, i = -1, \ldots, -d$, are the given initial conditions. Moreover, we assume that $\omega(t)$ is a scalar random white noise satisfying $\mathbf{E}[w(t)] = 0$, $\mathbf{E}[w(t)w(s)] = \sigma^2\delta_{ts}$. For convenience, we describe system (1) as $[A, B, C|d]$.

The objective of this paper is to explore the necessary and sufficient stabilizing conditions of system, $[A, B, C|d]$.

Before proceeding further, we introduce the following definition as follows.

Definition 1. System, $[A, B, C|d]$, is said to be asymptotically mean square stabilizable, if there exists a feedback control policy $u_{t-d} = K\hat{x}_{t|t-d-1}$, $t \geq d$, such that the following closed-loop system

$$x_{t+1} = Ax_t + [B + \omega(t)C]K\hat{x}_{t|t-d-1}, \tag{2}$$

is asymptotically mean square stable, i.e. for any initial values x_0 and u_i, $i = -1, \ldots, -d$, the state x_t satisfies $\lim_{t\to\infty} \mathbf{E}\|x_t\|^2 = 0$.

Remark 1. System, $[A, B, C|d]$, has a wide application in practice. To be specific, consider a wireless NCS as depicted in Fig. 1. The designed control signal is transmitted to the actuator through a lossy communication channel, where packet dropout and transmission delay occur simultaneously. In [15], the NCS can be described by:

$$x_{t+1} = Ax_t + \gamma_t Bu_{t-d}, \tag{3}$$

where $d > 0$ is the transmission delay and $\{\gamma_t\}_{t\geq 0}$ is an independent and identically distributed (i.i.d.) Bernoulli process representing the packet loss with probability distribution $\mathcal{P}(\gamma_t = 0) = p \in (0, 1)$. Denote $\omega_t = \gamma_t - \mathbf{E}[\gamma_t]$. Then, system (3) can be rewritten as:

$$x_{t+1} = Ax_t + (1 - p)Bu_{t-d} + w_t Bu_{t-d}, \tag{4}$$

which is a special case of system, $[A, B, C|d]$.

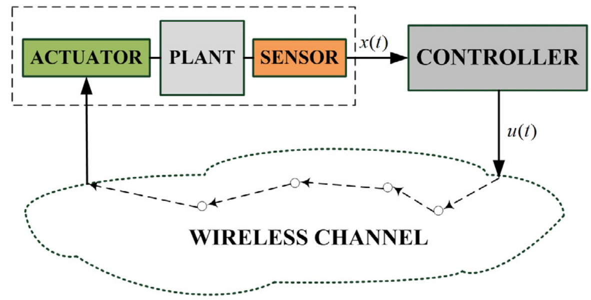

Fig. 1. NCS over wireless channel.

3. Main results

3.1. Necessary stabilizing condition

In this section, we propose a Lyapunov type necessary condition for stabilization of a more general stochastic system. Consider the following stochastic system with input delay and multiplicative noises:

$$x_{t+1} = Ax_t + Bu_{t-d} + \sum_{i=1}^{h} \omega_i(t)[\bar{A}_i x_t + \bar{B}_i u_{t-d}], \tag{5}$$

where $\mathbf{E}[w_i(t)] = 0$, $\mathbf{E}[w_i(t)w_j(s)] = \sigma_{ij}^2\delta_{ts}, i, j = 1, \ldots, h$. For any $t \geq 0$ and $1 \leq \tau \leq d + 1$, define $X_t = \mathbf{E}[x_t x_t']$ and $\hat{X}_{t|t-\tau} = \mathbf{E}[\hat{x}_{t|t-\tau}\hat{x}_{t|t-\tau}']$.

To begin with, we give the following lemma which is the basis to construct CLE.

Lemma 1. *Consider system (5) with $u_{t-d} = K\hat{x}_{t|t-d-1}$, $t \geq d$. Then, X_t and $\hat{X}_{t|t-\tau}$, $1 \leq \tau \leq d + 1$, satisfy the following difference equations:*

(a) For any $0 \leq t \leq d - 1$, we have

$$\begin{aligned} X_{t+1} = {} & AX_tA' + A\mathbf{E}[x_t u_{t-d}']B' + B\mathbf{E}[u_{t-d}x_t']A' \\ & + B\mathbf{E}[u_{t-d}u_{t-d}']B' + \sum_{i=1}^{h}\sum_{j=1}^{h} \sigma_{ij}^2 \Big(\bar{A}_i X_t \bar{A}_j' \\ & + \bar{A}_i \mathbf{E}[x_t u_{t-d}']\bar{B}_j' + \bar{B}_i \mathbf{E}[u_{t-d}x_t']\bar{A}_j' \\ & + \bar{B}_i \mathbf{E}[u_{t-d}u_{t-d}']\bar{B}_j'\Big), \end{aligned} \tag{6}$$

$$\begin{aligned} \hat{X}_{t+1|t-\tau} = {} & A\hat{X}_{t|t-\tau}A' + B\mathbf{E}[u_{t-d}u_{t-d}']B' \\ & + A\mathbf{E}[x_t u_{t-d}']B' + B\mathbf{E}[u_{t-d}x_t']A'. \end{aligned} \tag{7}$$

(b) For any $t \geq d$, we have

$$\begin{aligned} X_{t+1} = {} & A(X_t - \hat{X}_{t|t-d-1})A' \\ & + (A + BK)\hat{X}_{t|t-d-1}(A + BK)' \\ & + \sum_{i=1}^{h}\sum_{j=1}^{h} \sigma_{ij}^2 \Big(\bar{A}_i(X_t - \hat{X}_{t|t-d-1})\bar{A}_j' \\ & + (\bar{A}_i + \bar{B}_i K)\hat{X}_{t|t-d-1}(\bar{A}_j + \bar{B}_j K)'\Big), \end{aligned} \tag{8}$$

$$\begin{aligned} \hat{X}_{t+1|t-\tau} = {} & A(\hat{X}_{t|t-\tau} - \hat{X}_{t|t-d-1})A' \\ & + (A + BK)\hat{X}_{t|t-d-1}(A + BK)'. \end{aligned} \tag{9}$$

Proof. See Appendix A.

Next, with the help of Lemma 1, we are in a position to derive the Lyapunov type necessary condition and construct a CLE as follows.

Theorem 1. *Suppose that system (5) is stabilizable in the mean square sense. For any $Q \geq 0$, there exist matrices K and $H \geq P \geq 0$ satisfying the following CLE:*

$$\begin{aligned} H &= A(H-P)A' + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2\bar{A}_i(H-P)\bar{A}_j' \\ &\quad + Q + (A+BK)P(A+BK)' \\ &\quad + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2(\bar{A}_i+\bar{B}_iK)P(\bar{A}_j+\bar{B}_jK)', \end{aligned} \tag{10}$$

$$\begin{aligned} P &= A^dH(A^d)' - \sum_{t=0}^{d-1}A^{t+1}P(A')^{t+1} + \sum_{t=0}^{d-1}A^tQ(A')^t \\ &\quad + \sum_{t=0}^{d-1}A^t(A+BK)P(A+BK)'(A')^t. \end{aligned} \tag{11}$$

Moreover, if $Q > 0$, we have $H \geq P > 0$.

Proof. See Appendix B.

When $d = 0$, system (5) can be reduced to a delay free stochastic system with multiplicative noises and (10)–(11) in Theorem 1 can be rewritten as:

$$\begin{aligned} H &= A(H-P)A' + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2\bar{A}_i(H-P)\bar{A}_j' \\ &\quad + Q + (A+BK)P(A+BK)' \\ &\quad + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2(\bar{A}_i+\bar{B}_iK)P(\bar{A}_j+\bar{B}_jK)', \end{aligned} \tag{12}$$

$$P = H, \tag{13}$$

which is

$$\begin{aligned} P &= (A+BK)P(A+BK)' + Q \\ &\quad + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2(\bar{A}_i+\bar{B}_iK)P(\bar{A}_i+\bar{B}_jK)', \end{aligned} \tag{14}$$

It follows from Theorem 1 in [10] that the delay free stochastic system (5) is stabilizable in the mean square sense if and only if for any $Q > 0$, there exist matrices K and $P > 0$ satisfying (14). In this case, the feedback control policy $u_t = K\hat{x}_{t|t-d-1} = Kx_t$ is stabilizing.

When $w_i(t) = 0$, $i = 1, \ldots, h$, system (5) is reduced to a deterministic system $[A, B|d]$ with input delay and the CLE (10) in Theorem 1 can be rewritten as:

$$H - AHA' = Q - APA' + (A+BK)P(A+BK)', \tag{15}$$

which implies that

$$\begin{aligned} &\sum_{t=0}^{d-1}A^t(H-AHA')(A')^t = H - A^dH(A')^d \\ &= \sum_{t=0}^{d-1}A^t\big[Q - APA' + (A+BK)P(A+BK)'\big](A')^t. \end{aligned}$$

Compared with (11), we have $H = P$. In this case, the CLE (10)–(11) is equivalent to

$$-P + (A+BK)P(A+BK)' + Q = 0. \tag{16}$$

With the modified Smith predictor approach in [19], the deterministic system, $[A, B|d]$, is stabilizable in the mean square sense if and only if for any $Q > 0$, there exist matrices K and $P > 0$ satisfying (16).

Note that when system (5) is reduced to a delay free stochastic system or a deterministic delayed system, the reduced system is stabilizable in the mean square sense if and only if the CLE has a positive definite solution. In this case, Theorem 1 becomes the necessary and sufficient stabilizing condition. Unfortunately, how to utilize the CLE to guarantee the stabilization for the general stochastic system (5) remains open question.

3.2. Necessary and sufficient stabilizing condition

In this section, we explore the necessary and sufficient stabilizing conditions for system, $[A, B, C|d]$. It is interesting to find that the mean square stabilization of system, $[A, B, C|d]$, is equivalent to that of the following delay free stochastic system, (A, B, A^dC):

$$z_{t+1} = Az_t + Bv_t + \omega(t)A^dCv_t, \tag{17}$$

where v_t is designed to be the feedback of the state and $\omega(t)$ follows the same probability distribution in (1).

We are now in a position to propose the following Lyapunov stabilizing criterion.

Theorem 2. *The following statements are equivalent.*

(a) System, $[A, B, C|d]$, is stabilizable in the mean square sense.

(b) System, (A, B, A^dC), is stabilizable in the mean square sense.

(c) For any $Q > 0$, there exist matrices K and $P > 0$ satisfying the following DDLE:

$$\begin{aligned} P &= Q + (A+BK)P(A+BK)' \\ &\quad + \sigma^2A^dCKPK'C'(A')^d. \end{aligned} \tag{18}$$

(d) For any $Q > 0$, there exist matrices K and $P > 0$ satisfying the following DDLE:

$$\begin{aligned} P &= Q + (A+BK)'P(A+BK) \\ &\quad + \sigma^2K'C'(A')^dPA^dCK. \end{aligned} \tag{19}$$

(e) There exist matrices K and $P > 0$ such that:

$$\begin{aligned} P &> (A+BK)'P(A+BK) \\ &\quad + \sigma^2K'C'(A')^dPA^dCK. \end{aligned} \tag{20}$$

(f) There exist matrices Y and $S > 0$ such that:

$$\begin{bmatrix} -S & * & * \\ AS+BY & -S & * \\ \sigma A^dCY & 0 & -S \end{bmatrix} < 0, \tag{21}$$

where $*$ represents the corresponding transpose part. In this case, the stabilizing feedback gain is $K = YS^{-1}$.

Proof. See Appendix C.

In Theorem 2, a Lyapunov type stabilizing condition for system, $[A, B, C|d]$, is derived in terms of the positive definite solution to a DDLE, which is equivalent to the CLE in Theorem 1. Thus, the necessary condition in Theorem 1 is also sufficient for system, $[A, B, C|d]$. Moreover, compared with the CRE in [18], the Lyapunov stabilizing criterion in Theorem 2 can be verified by LMI solvers and is much convenient to use.

Next, we propose the spectrum stabilizing criterion. Define the following delay dependent Lyapunov operator $\mathcal{L}_K(\cdot)$ from $\mathbb{S}^n$ to $\mathbb{S}^n$:

$$\begin{aligned} \mathcal{L}_K(X) &= (A+BK)X(A+BK)' \\ &\quad + \sigma^2A^dCKXK'C'(A')^d, \ \forall X \in \mathbb{S}^n. \end{aligned} \tag{22}$$

Definition 2. The spectrum set of operator $\mathcal{L}_K(\cdot)$ is defined as:

$$\sigma(\mathcal{L}_K) = \{\lambda \in \mathbb{C} : \mathcal{L}_K(\hat{X}) = \lambda\hat{X}, \hat{X} \in \mathbb{S}^n, \hat{X} \neq 0\}. \tag{23}$$

The spectral radius of $\mathcal{L}_K(\cdot)$ is defined as:

$$\rho(\mathcal{L}_K) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{L}_K)\}. \tag{24}$$

Corollary 1. *System, $[A, B, C|d]$, is stabilizable in the mean square sense if and only if there exists a feedback gain matrix K such that $\rho(\mathcal{L}_K) < 1$.*

Proof. The proof can be derived directly by applying Theorem 1 in [9] and Theorem 2.

4. Illustrative numerical examples

In this section, we will use two examples to demonstrate the effectiveness of our theoretic results.

Example 1. Consider the following discrete time stochastic system (5) with $\sigma_1^2 = 1$, $d_1 = 1$ and

$$A_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & -\frac{1}{2} \end{bmatrix},$$
$$\bar{A}_1 = \begin{bmatrix} \frac{3}{4} & -1 \\ 0 & \frac{1}{4} \end{bmatrix}, \; \bar{B}_1 = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{bmatrix}.$$

It is easy to verify that there exist

$$K_1 = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.25 \end{bmatrix}, \; Q_1 = \begin{bmatrix} 1 & 0 \\ 0.25 & 0.25 \end{bmatrix} > 0.$$

satisfying

$$Q_1 \geq A_1' Q_1 A_1 + \bar{A}_1' Q_1 \bar{A}_1, \tag{25}$$
$$Q_1 > (A_1 + B_1 K_1)' Q_1 (A_1 + B_1 K_1) + (\bar{A}_1 + \bar{B}_1 K_1)' Q_1 (\bar{A}_1 + \bar{B}_1 K_1). \tag{26}$$

In this case, define the Lyapunov function

$$L_1(x_t) = \mathbf{E}[x_t' Q_1 x_t] = \mathbf{Tr}(Q_1 X_t). \tag{27}$$

For any $t \geq d_1$, with $u_{t-d_1} = K_1 \hat{x}_{t|t-d_1-1}$, it follows from Lemma 1 that

$$\begin{aligned} & L_1(x_{t+1}) - L_1(x_t) = \mathbf{Tr}(Q_1 X_{t+1}) - \mathbf{Tr}(Q_1 X_t) \\ = & Tr\Big\{(-Q_1 + A_1' Q_1 A_1 + \bar{A}_1' Q_1 \bar{A}_1)(X_t - \hat{X}_{t|t-d_1-1}) \\ & + \big[-Q_1 + (A_1 + B_1 K_1)' Q_1 (A_1 + B_1 K_1) \\ & + (\bar{A}_1 + \bar{B}_1 K_1)' Q_1 (\bar{A}_1 + \bar{B}_1 K_1)\big] \hat{X}_{t|t-d_1-1}\Big\} < 0, \end{aligned}$$

which yields that system, $[A_1, B_1; \bar{A}_1, \bar{B}_1 | d_1]$, is stabilizable. By Theorem 1, the CLE (10)–(11) has the following positive definite solution

$$H_1 = \begin{bmatrix} 13.0682 & -0.2964 \\ -0.2964 & 1.1474 \end{bmatrix} > 0, \; P_1 = \begin{bmatrix} 3.6139 & 0.0403 \\ 0.0403 & 1.0778 \end{bmatrix} > 0.$$

Example 2. Consider the following system, $[A_2, B_2, C_2|d_2]$, with $\sigma_2^2 = 1$, $d_2 = 2$ and $A_2 = \frac{5}{4}$, $B_2 = \frac{1}{2}$, $C_2 = \frac{1}{3}$. By utilizing MATLAB, we can give a simple path of $w(t)$ as shown in Fig. 2.

Suppose the initial conditions are given with $x_0 = 2$ and $u_{-1} = u_{-2} = 0$. It follows from Lemma 1 that

$$X_1 = A_2 X_0 A_2' = 6.25, \; X_2 = A_2 X_1 A_2' = 9.7656.$$

Then, by solving (21) in Theorem 2, we find an admissible solution that $S_2 = \frac{1}{4}$ and $Y_2 = -\frac{1}{4}$, which implies that $P_2 = S_2^{-1} = 4$ and $K_2 = Y_2 P_2 = -1$. By Theorem 2, we obtain that $u_2 = -\hat{x}_{t|t-3}$ is the stabilizing control policy. The simulation of $\mathbf{E}[x_t^2]$ is shown in Fig. 3.

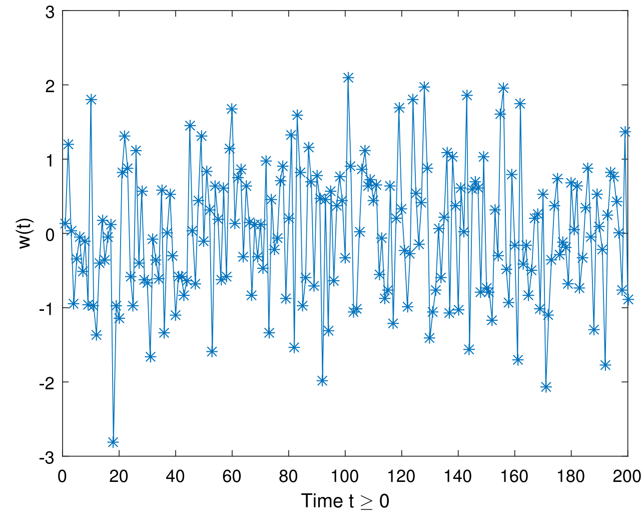

Fig. 2. A simple path of $w(t)$.

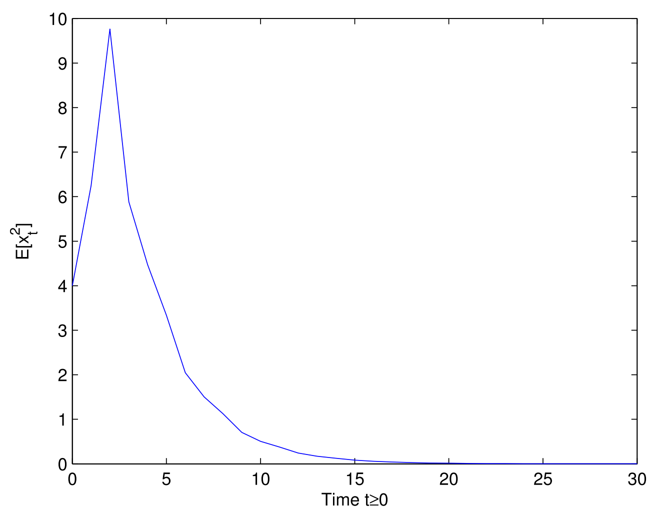

Fig. 3. Simulation of $\mathbf{E}[x_t^2]$.

5. Conclusion

In this paper, we developed a DDLE based approach to study the mean square stabilization problem for stochastic system with input delay and control dependent noise. We proposed the necessary and sufficient condition for stabilization in terms of a positive definite solution to a DDLE, which is in parallel with the classical results in stochastic control. However, how to utilize the DDLE to guarantee the stabilization of a general stochastic time delay system with state and control dependent noise remains open question, which defines a promising future research direction.

Appendix A. Proof of Lemma 1

Proof. For $X_t = \mathbf{E}[x_t x_t']$, it follows from (5) that

$$\begin{aligned} X_{t+1} = & A X_t A' + A\mathbf{E}[x_t u_{t-d}'] B' + B\mathbf{E}[u_{t-d} x_t'] A' \\ & + B\mathbf{E}[u_{t-d} u_{t-d}'] B' + \sum_{i=1}^{h} \sum_{j=1}^{h} \sigma_{ij}^2 \Big(\bar{A}_i X_t \bar{A}_j' \\ & + \bar{A}_i \mathbf{E}[x_t u_{t-d}'] \bar{B}_j' + \bar{B}_i \mathbf{E}[u_{t-d} x_t'] \bar{A}_j' \\ & + \bar{B}_i \mathbf{E}[u_{t-d} u_{t-d}'] \bar{B}_j' \Big), \end{aligned} \tag{A.1}$$

which implies that (6) holds for $0 \le t \le d-1$. For any $t \ge d$, with the feedback control policy $u_{t-d} = K\hat{x}_{t|t-d-1}$, (6) can be equivalently rewritten as

$$\begin{aligned} & X_{t+1} \\ &= AX_tA' + A\mathbf{E}[x_t x'_{t|t-d-1}]K'B' + BK\mathbf{E}[x_{t|t-d-1}x'_t] \\ &\quad \times A' + BK\mathbf{E}[x_{t|t-d-1}x'_{t|t-d-1}]K'B' + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2 \\ &\quad \times \Big(\bar{A}_iX_t\bar{A}'_j + \bar{B}_i\mathbf{E}[x_{t|t-d-1}x'_{t|t-d-1}]\bar{B}'_j \\ &\quad + \bar{A}_i\mathbf{E}[x_t x'_{t|t-d-1}]\bar{B}'_j + \bar{B}_i\mathbf{E}[x_{t|t-d-1}x'_t]\bar{A}'_j\Big) \\ &= A(X_t - \hat{X}_{t|t-d-1})A' + (A+BK)\hat{X}_{t|t-d-1} \\ &\quad \times (A+BK)' + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2\Big(\bar{A}_i(X_t - \hat{X}_{t|t-d-1}) \\ &\quad \times \bar{A}'_j + (\bar{A}_i + \bar{B}_iK)\hat{X}_{t|t-d-1}(\bar{A}_j + \bar{B}_jK)'\Big), \end{aligned} \tag{A.2}$$

where $\mathbf{E}[\hat{x}_{t|t-d-1}x'_t] = \mathbf{E}[\hat{x}_{t|t-d-1}\hat{x}'_{t|t-d-1}] = \hat{X}_{t|t-d-1}$ can be used to derive the last equality. Moreover, due to the fact that x_t is measurable w.r.t. the filter $\mathscr{F}_{t-1}$, we obtain $\hat{x}_{t|t-1} = x_t$, $\hat{X}_{t|t-1} = X_t$. In this case, for any $0 \le t \le d-1$, by taking the conditional expectation w.r.t. $\mathscr{F}_{t-\tau}$, $1 \le \tau \le d+1$, on both side of (5), we have $\hat{x}_{t+1|t-\tau} = A\hat{x}_{t|t-\tau} + Bu_{t-d}$, which implies that

$$\begin{aligned} \hat{X}_{t+1|t-\tau} &= A\hat{X}_{t|t-\tau}A' + B\mathbf{E}[u_{t-d}u'_{t-d}]B' \\ &\quad + A\mathbf{E}[x_t u'_{t-d}]B' + B\mathbf{E}[u_{t-d}x'_t]A'. \end{aligned} \tag{A.3}$$

Similarly, for $t \ge d$, we have $\hat{x}_{t+1|t-\tau} = A\hat{x}_{t|t-\tau} + BK\hat{x}_{t|t-d-1}$. It follows that

$$\begin{aligned} \hat{X}_{t+1|t-\tau} &= A(\hat{X}_{t|t-\tau} - \hat{X}_{t|t-d-1})A' \\ &\quad + (A+BK)\hat{X}_{t|t-d-1}(A+BK)', \end{aligned} \tag{A.4}$$

which completes the proof.

Appendix B. Proof of Theorem 1

Proof. When system (5) is stabilizable in the mean square sense, by Lemma 1, there exists a stabilizing control policy $u_{t-d} = K\hat{x}_{t|t-d-1}$, $t \ge d$, such that $\lim_{t\to\infty} X_t = 0$. For any $1 \le \tau \le d+1$, denote $\tilde{x}_{t|t-\tau} = x_t - \hat{x}_{t|t-\tau}$, which is orthogonal to $\hat{x}_{t|t-\tau}$. It follows that

$$\begin{aligned} X_t &= \mathbf{E}\big[(\hat{x}_{t|t-\tau} + \tilde{x}_{t|t-\tau})(\hat{x}_{t|t-\tau} + \tilde{x}_{t|t-\tau})'\big] \\ &= \hat{X}_{t|t-\tau} + \mathbf{E}[\tilde{x}_{t|t-\tau}\tilde{x}'_{t|t-\tau}], \end{aligned} \tag{B.1}$$

which implies that $0 \le \hat{X}_{t|t-\tau} \le X_t$. When system (5) is stabilizable, we have $\lim_{t\to\infty}\hat{X}_{t|t-\tau} = 0$. For any $0 \le t \le d-1$, we have $\hat{x}_{t|t-\tau} = \mathbf{E}[x_t]$.

Let the initial conditions be $X_0 = Q \ge 0$ and $u_{t-d} = K\hat{x}_{t|t-d-1}$, $0 \le t \le d-1$. It follows from Lemma 1 that (8)–(9) hold for any $t \ge 0$. To construct the CLE (10)–(11), denote $H = \sum_{t=0}^{\infty} X_t$ and $P = \sum_{t=0}^{\infty}\hat{X}_{t|t-d-1}$. By Theorem 1 in [8], the stabilization of system (5) guarantees the existence of H and P. Moreover, we have $0 \le P \le H < \infty$. Then, it follows from (8) that

$$\begin{aligned} -Q &= \lim_{t\to\infty} X_t - X_0 = \sum_{t=0}^{\infty}(X_{t+1} - X_t) \\ &= \sum_{t=0}^{\infty}\Big[-X_t + A(X_t - \hat{X}_{t|t-d-1})A' \\ &\quad + (A+BK)\hat{X}_{t|t-d-1}(A+BK)' \\ &\quad + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2\big(\bar{A}_i(X_t - \hat{X}_{t|t-d-1})\bar{A}'_j \\ &\quad + (\bar{A}_i + \bar{B}_iK)\hat{X}_{t|t-d-1}(\bar{A}_j + \bar{B}_jK)'\big)\Big] \\ &= -H + A(H-P)A' + (A+BK)P \\ &\quad \times (A+BK)' + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2\bar{A}_i(H-P)\bar{A}'_j \\ &\quad + \sum_{i=1}^{h}\sum_{j=1}^{h}\sigma_{ij}^2(\bar{A}_i + \bar{B}_iK)P(\bar{A}_j + \bar{B}_jK)', \end{aligned} \tag{B.2}$$

which is equivalent to (10).

On the other hand, for any $t < 0$, denote $\hat{X}_{t|t-\tau} = 0$. For any $0 \le t \le d-1$, it follows from (7) that

$$\hat{X}_{t|t-d-1} = (A+BK)^tQ((A+BK)')^t. \tag{B.3}$$

For any $t \ge d$, it follows from (9) that

$$\begin{aligned} \hat{X}_{t|t-d-1} &= A\hat{X}_{t-1|t-d-1}A' + (A+BK)\hat{X}_{t-1|t-d-1} \\ &\quad \times (A+BK)' - A\hat{X}_{t-1|t-d-2}A' \\ &= A^dX_{t-d}(A')^d + \sum_{i=0}^{d-1}A^i(A+BK) \\ &\quad \times \hat{X}_{t-i-1|t-i-d-2}(A+BK)'(A')^i \\ &\quad - \sum_{i=0}^{d-1}A^{i+1}\hat{X}_{t-i-1|t-i-d-2}(A')^{i+1}. \end{aligned} \tag{B.4}$$

Then, by taking the sum from $t = 0$ to infinity on both side of (B.4), we obtain

$$\begin{aligned} &\sum_{t=0}^{\infty}\hat{X}_{t|t-d-1} = \sum_{t=0}^{d-1}\hat{X}_{t|t-d-1} + \sum_{t=d}^{\infty}\hat{X}_{t|t-d-1} \\ &= \sum_{t=0}^{d-1}(A+BK)^tQ((A+BK)')^t + A^d\sum_{t=0}^{\infty}X_t(A')^d \\ &\quad + \sum_{i=0}^{d-1}A^i(A+BK)\Big(\sum_{t=d}^{\infty}\hat{X}_{t-i-1|t-i-d-2}\Big)(A+BK)'(A')^i \\ &\quad - \sum_{i=0}^{d-1}A^{i+1}\Big(\sum_{t=d}^{\infty}\hat{X}_{t-i-1|t-i-d-2}\Big)(A')^{i+1}. \end{aligned} \tag{B.5}$$

It is equivalent to

$$\begin{aligned} P &= A^dH(A')^d + \sum_{t=0}^{d-1}A^t(A+BK)P(A+BK)'(A')^t \\ &\quad - \sum_{t=0}^{d-1}A^{t+1}\hat{P}(A')^{t+1} + L, \end{aligned} \tag{B.6}$$

where

$$\begin{aligned} L &= \sum_{i=0}^{d-1}(A+BK)^iQ((A+BK)')^i - \sum_{i=0}^{d-2}A^i \\ &\quad \times \Big[\sum_{t=0}^{d-i-2}(A+BK)^{t+1}Q((A+BK)')^{t+1}\Big](A')^i \\ &\quad + \sum_{i=0}^{d-2}A^{i+1}\Big[\sum_{t=0}^{d-i-2}(A+BK)^tQ((A+BK)')^t\Big](A')^{i+1} \\ &= Q + \sum_{t=0}^{d-2}A(A+BK)^tQ((A+BK)')^tA' \end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{d-2} A^i \Big[\sum_{t=0}^{d-i-2} A(A+BK)^t Q((A+BK)')^t \\
&\times K'B' + BK(A+BK)^t Q((A+BK)')^t A' \\
&+ BK(A+BK)^t Q((A+BK)')^t K'B' \Big] (A')^i.
\end{aligned}
$$

It follows that $L = \sum_{t=0}^{d-1} A^t Q(A')^t$. Therefore, we have (11) holds. Moreover, if $X_0 = Q > 0$, it follows that

$$
H = \sum_{t=0}^{\infty} X_t \geq P = \sum_{t=0}^{\infty} \hat{X}_{t|t-d-1} \geq Q > 0, \tag{B.7}
$$

which completes the proof.

Appendix C. Proof of Theorem 2

Proof. Similar to Theorem 1 in [10], we have that $(c) \Leftrightarrow (d)$ and $(d) \Leftrightarrow (e)$. For the sake of simplicity, we prove that $(a) \Rightarrow (c)$, $(d) \Rightarrow (b)$, $(a) \Leftrightarrow (b)$, and $(e) \Leftrightarrow (f)$.

$(a) \Rightarrow (c)$. Suppose system, $[A, B, C|d]$, is stabilizable in the mean square sense. By Theorem 1, for any $Q > 0$, there exist matrices K and $H \geq P > 0$ satisfying the following CLE:

$$
\begin{aligned}
H &= A(H-P)A' + (A+BK)P(A+BK)' \\
&\quad + \sigma^2 CKPK'C' + Q,
\end{aligned} \tag{C.1}
$$

$$
\begin{aligned}
P &= A^d H(A^d)' - \sum_{t=0}^{d-1} A^{t+1} P(A')^{t+1} + \sum_{t=0}^{d-1} A^t Q(A')^t \\
&\quad + \sum_{t=0}^{d-1} A^t (A+BK)P(A+BK)'(A')^t.
\end{aligned} \tag{C.2}
$$

Next, we show that the CLE (C.1)–(C.2) is equivalent to the DDLE (18).

It follows from (C.1) that

$$
\begin{aligned}
&H - A^d H(A')^d = \sum_{t=0}^{d-1} A^t (H - AHA')(A')^d \\
&= \sum_{t=0}^{d-1} A^t \big[-APA' + (A+BK)P(A+BK)' \\
&\quad + \sigma^2 CKPK'C' + Q \big] (A')^t.
\end{aligned} \tag{C.3}
$$

Compared with (C.2), we have

$$
H = P + \sigma^2 \sum_{t=0}^{d-1} A^t CKPK'C'(A')^t. \tag{C.4}
$$

By applying (C.4) in (C.1), we have

$$
\begin{aligned}
&P + \sigma^2 \sum_{t=0}^{d-1} A^t CKPK'C'(A')^t \\
&= APA' + \sigma^2 \sum_{t=0}^{d-1} A^{t+1} CKPK'C'(A')^{t+1} \\
&\quad - APA' + (A+BK)P(A+BK)' \\
&\quad + \sigma^2 CKPK'C' + Q,
\end{aligned} \tag{C.5}
$$

which is equivalent to the DDLE (18).

$(d) \Rightarrow (b)$. Suppose there exist matrices K and $P > 0$ satisfying the DDLE (19) with $Q > 0$. With the state feedback control policy $v_t = Kz_t$, system, $(A, B, A^d C)$, can be rewritten as:

$$
z_{t+1} = (A+BK)z_t + \omega(t) A^d CKz_t. \tag{C.6}
$$

Define the following Lyapunov function:

$$
L_t(z_t) = \mathbf{E}[z_t' P z_t] > 0. \tag{C.7}
$$

Then, we have

$$
\begin{aligned}
&L_{t+1}(z_{t+1}) - L_t(z_t) \\
&= \mathbf{E}\Big[\big((A+BK)z_t + \omega(t) A^d CKz_t \big)' P \big((A+BK)z_t \\
&\quad + \omega(t) A^d CKz_t \big) \Big] - \mathbf{E}[z_t' P z_t] \\
&= \mathbf{E}\Big[z_t' \big(-P + (A+BK)'P(A+BK) \\
&\quad + \sigma^2 K'C'(A')^d P A^d CK \big) z_t \Big] \\
&= -\mathbf{E}[z_t' Q z_t] < 0.
\end{aligned} \tag{C.8}
$$

By means of the Lyapunov stability theory, system, $(A, B, A^d C)$, is stabilizable and the control policy $v_t = Kz_t$ is stabilizing.

$(a) \Leftrightarrow (b)$. It follows from Theorem 3 of [18], system, $[A, B, C|d]$, is stabilizable in the mean square sense if and only if the following CRE:

$$
Z = A'ZA + Q - A'ZBL^{-1}B'ZA, \tag{C.9}
$$

$$
X = Z + \sum_{i=0}^{d-1} (A')^{i+1} ZBL^{-1}B'ZA^{i+1}, \tag{C.10}
$$

has a unique positive definite solution $Z > 0$, where

$$
L = B'ZB + \sigma^2 C'XC + R, \; Q > 0, \; R > 0. \tag{C.11}
$$

By (C.9), we obtain

$$
-Z + A'ZA + Q = A'ZBL^{-1}B'ZA \geq 0. \tag{C.12}
$$

Applying (C.12) in (C.10) leads to

$$
\begin{aligned}
X &= Z + \sum_{i=0}^{d-1} (A')^i (-Z + A'ZA + Q) A^i \\
&= (A')^d Z A^d + \sum_{i=0}^{d-1} (A')^i Q A^i.
\end{aligned} \tag{C.13}
$$

In this case, the parameter L in (C.11) can be rewritten as

$$
\begin{aligned}
L &= R + B'ZB + \sigma^2 C'(A')^d Z A^d C \\
&\quad + \sigma^2 \sum_{i=0}^{d-1} C'(A')^i Q A^i C,
\end{aligned} \tag{C.14}
$$

Therefore, the coupled Riccati equation (C.9)–(C.10) is equivalent to the following GARE:

$$
\begin{aligned}
Z &= A'ZA + Q - A'ZB \Big(B'ZB + \sigma^2 C'(A')^d Z A^d C \\
&\quad + \sigma^2 \sum_{i=0}^{d-1} C'(A')^i Q A^i C + R \Big)^{-1} B'ZA.
\end{aligned} \tag{C.15}
$$

On the other hand, for system, $(A, B, A^d C)$, consider the following infinite horizon LQ optimization problem:

$$
J(v_t) = \sum_{t=0}^{\infty} \mathbf{E}[z_t' Q z_t + v_t' R_0 v_t], \tag{C.16}
$$

where $Q > 0$, $R_0 = \sigma^2 \sum_{i=0}^{d-1} B'(A')^i Q A^i B + R > 0$. It follows from [8], system, $(A, B, A^d C)$ is stabilizable in the mean square sense if and only if the following GARE:

$$
\begin{aligned}
P &= A'PA + Q - A'PB \big(R_1 + B'PB \\
&\quad + \sigma^2 C'(A')^d P A^d C \big)^{-1} B'PA,
\end{aligned} \tag{C.17}
$$

has a positive definite solution $P > 0$. When $P = Z$, it is evident that the GARE (C.17) is equivalent to (C.15), which indicates that the mean square stabilization of $[A, B, C|d]$ is equivalent to that of system $(A, B, A^d C)$.

$(e) \Leftrightarrow (f)$. Suppose that there exist matrices K and $P > 0$ satisfying (20). By utilizing Schur complement decomposition in [20], we have

$$S = \begin{bmatrix} -P & * & * \\ A + BK & -P^{-1} & * \\ \sigma A^d CK & 0 & -P^{-1} \end{bmatrix} < 0, \tag{C.18}$$

where $*$ represents the corresponding transpose part. Then, the matrix transformation technique gives that

$$\begin{aligned} \Lambda &= \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} S \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} -P^{-1} & * & * \\ AP^{-1} + BKP^{-1} & -P^{-1} & * \\ \sigma A^d CKP^{-1} & 0 & -P^{-1} \end{bmatrix} < 0, \end{aligned} \tag{C.19}$$

which is equivalent to (21) with $S = P^{-1} > 0$ and $Y = KP^{-1}$. Conversely, if there exist matrices Y and $S > 0$ satisfying (21), it is easy to verify that (20) holds for $P = S^{-1} > 0$ and $K = YS^{-1}$.

References

[1] H.J. Kushner, Stochastic Stability and Control, Academic, New York, 1967.
[2] R.Z. Hasminskii, Stochastic Stability of Differential Equations, Sijthoff and Noordhoff, Alphen, 1980.
[3] B. Øksendal, Stochastic Differential Equations: An Introduction with Application, sixth ed., Springer, New York, 2003.
[4] J.B. Moore, X. Zhou, A.E.B. Lim, Discrete time LQG controls with control dependent noise, Syst. Control Lett. 36 (3) (1999) 199–206.
[5] M.A. Rami, X. Chen, J.B. Moore, X. Zhou, Solvability and asymptotic behavior of generalized Riccati equations arising in indefinite stochastic LQ controls, IEEE Trans. Automat. Control 46 (3) (2001) 428–440.
[6] W. Zhang, B.S. Chen, On stabilizability and exact observability of stochastic systems with their applications, Automatica 40 (1) (2004) 87–94.
[7] W. Zhang, H. Zhang, B.S. Chen, Generalized Lyapunov equation approach to state-dependent stochastic stabilization/detectability criterion, IEEE Trans. Automat. Control 53 (7) (2008) 1630–1642.
[8] Y. Huang, W. Zhang, H. Zhang, Infinite horizon linear quadratic optimal control for discrete-time stochastic systems, Asian J. Control 10 (5) (2008) 608–615.
[9] T. Hou, W. Zhang, Study on general stability and stabilizability of linear discrete time stochastic systems, Asian J. Control 13 (6) (2011) 977–987.
[10] M.A. Rami, X. Zhou, Linear matrix inequalities, Riccati equations and indefinite stochastic linear quadratic controls, IEEE Trans. Automat. Control 45 (6) (2000) 1131–1143.
[11] W.S. Wong, R.W. Brockett, Systems with finite communication bandwidth constraints II: Stabilization with limited information feedback, IEEE Trans. Automat. Control 44 (5) (1999) 1049–1053.
[12] J.P. Hespanha, P. Naghshtabrizi, Y. Xu, A survey of recent results in networked control systems, Proc. IEEE 95 (1) (2007) 138–162.
[13] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, S. Sastry, Foundations of control and estimation over lossy networks, Proc. IEEE 95 (1) (2007) 163–187.
[14] N. Xiao, L. Xie, L. Qiu, Feedback stabilization of discrete-time networked systems over fading channels, IEEE Trans. Automat. Control 57 (9) (2012) 2176–2189.
[15] C. Tan, L. Li, H. Zhang, Stabilization of networked control systems with network-induced delay and packet dropout, Automatica 59 (2015) 194–199.
[16] C. Tan, H. Zhang, Necessary and sufficient stabilizing conditions for networked control systems with simultaneous transmission delay and packet dropout, IEEE Trans. Automat. Control 62 (8) (2017) 4011–4016.
[17] C. Tan, H. Zhang, W.S. Wong, Delay-dependent algebraic Riccati equation to stabilization of networked control systems: continuous-time case, IEEE Trans. Cybern. 48 (10) (2018) 2783–2794.
[18] H. Zhang, L. Li, J. Xu, M. Fu, Linear quadratic regulation and stabilization of discrete-time systems with delay and multiplicative noise, IEEE Trans. Automat. Control 60 (10) (2015) 2599–2613.
[19] K. Watanabe, M. Ito, A process-model control for linear systems with delay, IEEE Trans. Automat. Control 26 (6) (1981) 1261–1269.
[20] S. Boyd, L. El Ghaoui, E. Feron, V. balakrishnan, Linear Matrix Inequality in Systems and Control Theory, SIAM, PA, Philadelphia, 1994.